4.3 – Spanning Sets

Definition: (generalizes linear combination from Section 3.1)
If $\mathbf{w}$ is a vector in a vector space V, then $\mathbf{w}$ is said to be a **linear combination** of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in V if $\mathbf{w}$ can be expressed in the form $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \ldots + k_r\mathbf{v}_r$, where each k_i is a scalar. The scalars are called the **coefficients** of the linear combination. In the case where $r = 1$, we have $\mathbf{w} = k_1\mathbf{v}_1$, in which case the linear combination is a scalar multiple of the vector.

$(-2, 1, -2)$

#6 In each part, express the vector as a linear combination of of $\mathbf{p}_1 = 2 + x + 4x^2$, $\mathbf{p}_2 = 1 - x + 3x^2$, and $\mathbf{p}_3 = 3 + 2x + 5x^2$.

a. $-9 - 7x - 15x^2 = \vec{p}$

b. $6 + 11x + 6x^2$

c. 0

d. $7 + 8x + 9x^2$

$$\vec{p}_1 = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

a) We want $k_1, k_2, k_3 \ni k_1\vec{p}_1 + k_2\vec{p}_2 + k_3\vec{p}_3 = \vec{p}$

$$k_1(2+x+4x^2) + k_2(1-x+3x^2) + k_3(3+2x+5x^2) = -9-7x-15x^2$$

$$2k+k_2+3k_3 = -9$$

$$k_1-k_2+2k_3 = -7$$

and so on

We can use: $\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 3 & 5 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \vec{p} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & b_1 \\ 1 & -1 & 2 & b_2 \\ 4 & 3 & 5 & b_3 \end{array}\right] \to \begin{array}{cccc} k_1 & k_2 & k_3 & \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2}b_1 + 2b_2 + \frac{5}{2}b_3 \\ 0 & 1 & 0 & \frac{3}{2}b_1 - b_2 - \frac{1}{2}b_3 \\ 0 & 0 & 1 & \frac{7}{2}b_1 - b_2 - \frac{3}{2}b_3 \end{array}\right]$$

For a) if $b_1 = -9$, $b_2 = -7$, $b_3 = -15$, then
$k_1 = -2$, $k_2 = 1$, $k_3 = -2$

check: $-2\vec{p}_1 + \vec{P}_2 - 2\vec{P}_3$

and likewise for (b) – (d).

We note that there are no restrictions on b_1, b_2, b_3. Thus we can always find a linear combination that yields whatever vector we want in P_2.

We then say that $\{\vec{P}_1, \vec{P}_2, \vec{P}_3\}$ spans P_2.

We can also that span$\{\vec{P}_1, \vec{P}_2, \vec{P}_3\} = P_2$.

Definition: The **span** of a nonempty set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ of vectors is the set of all possible linear combinations of vectors in S, denoted by span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ or span(S). If $W = $ span(S), then we say that W is **spanned** by S.

#9. Determine whether the following polynomials span P_2.

$\vec{p}_1 = 1 - x + 2x^2$, $\vec{p}_2 = 3 + x$, $\vec{p}_3 = 5 - x + 4x^2$,
$\vec{p}_4 = -2 - 2x + 2x^2$.

That is, can we build every polynomial of deg ≤ 2 using these?

That is, does $k_1 \vec{p}_1 + k_2 \vec{p}_2 + k_3 \vec{p}_3 + k_4 \vec{p}_4 = \vec{p}$ always have a solution?

$$\left[\begin{array}{cccc|c} 1 & 3 & 5 & -2 & b_1 \\ -1 & 1 & -1 & -2 & b_2 \\ 2 & 0 & 4 & 2 & b_3 \end{array}\right] \xrightarrow[\text{reduce}]{\text{row}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & \frac{1}{4}b_1 - \frac{3}{4}b_2 \\ 0 & 1 & 1 & -1 & \frac{1}{4}b_1 + \frac{1}{4}b_2 \\ 0 & 0 & 0 & 0 & b_1 - 3b_2 - 2b_3 \end{array}\right]$$

This has a solution iff $b_1 - 3b_2 - 2b_3 = 0$.

The polynomials do not span P_2.

#12 Let $T_A : R^2 \to R^2$ be multiplication by A. Determine whether the vector $\mathbf{u} = (1, 2)$ is in the span of $\{T_A(\mathbf{e}_1), T_A(\mathbf{e}_2)\}$. — can we build $(1, 2)$ as $k_1 T(\vec{e}_1) + k_2 T(\vec{e}_2)$?

a. $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ b. $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

a) $T_A(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T_A(\vec{e}_2) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Solve $k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 2 \end{array}\right]$

b). No.

#16 Let W be the solution space to the system $A\mathbf{x} = \mathbf{0}$. Determine whether the set $\{\mathbf{u}, \mathbf{v}\}$ spans W.

$A = \left[\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 0 \\ 0 & 2 & -2 & 2 & 0 \\ 0 & 3 & -3 & 3 & 0 \end{array}\right] \xrightarrow[\text{reduce}]{\text{row}} \begin{bmatrix} 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $x_2 - x_3 + x_4 = 0$

a. $\mathbf{u} = (1, 1, 1, 0)$, $\mathbf{v} = (0, -1, 0, 1)$

b. $\mathbf{u} = (0, 1, 1, 0)$, $\mathbf{v} = (1, 0, 1, 1)$

$x_1 = r$ (free)

$x_3 = s$ (free)

$x_4 = t$ (free)

$x_2 = s - t \Rightarrow \vec{x} = \begin{bmatrix} r \\ s-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} t$

Solution

The solution space is spanned by $\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{w}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{w}_2}, \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{w}_3} \right\}$.

We can see that $\vec{w}_3 = \vec{v}$

Do there exist a, b such that $a\vec{u} + b\vec{v} = \vec{w}_1$? $\vec{w}_2$?

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \leftarrow 0 = 1$$

No.

Likewise, (6) : No.

Ex: $\vec{w}_1 = (2,1,3)$, $\vec{w}_2 = (5,6,7)$

$a(2,1,3) + b(5,6,7)$

Theorem 4.3.1 If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then:
a) The set W of all possible linear combinations of the vectors in S is a subspace of V.
b) The set W in part (a) is the "smallest" subspace of V that contains all of the vectors in S in the sense that any other subspace that contains those vectors contains W.

Theorem 4.3.2 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ are nonempty sets of vectors in a vector space V, then $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ if and only if each vector in S is a linear combination of those in S', and each vector in S' is a linear combination of those in S.

#20 Let $\mathbf{v}_1 = (1,\ 6,\ 4)$, $\mathbf{v}_2 = (2,\ 4,\ -1)$, $\mathbf{v}_3 = (-1,\ 2,\ 5)$, and $\mathbf{w}_1 = (1,\ -2,\ -5)$, $\mathbf{w}_2 = (0,\ 8,\ 9)$. Use Theorem 4.3.2 to show that $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}$.

Show we can use $\{\vec{w}_1, \vec{w}_2\}$ to make $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$\vec{v}_1$: $a\vec{w}_1 + b\vec{w}_2 = \vec{v}_1 \Rightarrow \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} a + \begin{bmatrix} 0 \\ 8 \\ 9 \end{bmatrix} b = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$

$$\begin{array}{c} \begin{array}{ccccc} \vec{w}_1 & \vec{w}_2 & \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \\ \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & -1 \\ -2 & 8 & 6 & 4 & 2 \\ -5 & 9 & 4 & -1 & 5 \end{array}\right] \end{array} \rightarrow \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so $\vec{v}_1, \vec{v}_2$ & $\vec{v}_3 \in \operatorname{span}\{\vec{w}_1, \vec{w}_2\}$

$$\begin{array}{c} \begin{array}{ccccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{w}_1 & \vec{w}_2 \end{array} \\ \left[\begin{array}{ccc|cc} 1 & 2 & -1 & 1 & 0 \\ 6 & 4 & 2 & -2 & 8 \\ 4 & -1 & 5 & -5 & 9 \end{array}\right] \end{array} \rightarrow \left[\begin{array}{ccc|cc} 1 & 0 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$